Math 220 Problems

1. Let $\vec{a}, \vec{b}, \vec{c}$ be three 3-D vectors: $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, etc. Each vector can be identifyed with a column matrix, for instance

$$\vec{a} \longleftrightarrow A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Based on this correspondence, prove the following:
a) the scalar product $\vec{a} \cdot \vec{b}$ is equal to $A^T \cdot B$, where A, B are the columns associated to the two vectors;
b) the vector product $\vec{b} \times \vec{c}$ is associated to the column

$$\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

c) the mixed product $\vec{a} \cdot (\vec{b} \times \vec{c})$ is equal to the determinant of the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

d) The mixed product of any three vectors is zero if any two of them are parallel to each other;
e)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = -\vec{a} \cdot (\vec{c} \times \vec{b}).$$

2. Consider the 2×2 real matrix

$$M = \begin{bmatrix} \frac{1+a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{1+a}{2} \end{bmatrix},$$

where $a \in [0, 1)$.
a) Compute the determinant of M.
b) Compute the matrix $L = \lim_{n\to\infty} M^n$ and its determinant.
c) Verify whether $det(L) = \lim_{n\to\infty} (detM)^n$.
d) Assume that the matrix M describes the transition probabilities for a particle in a two-state system, such as a particle diffusing from container A to container B through a permeable wall. If at time $t_n = n$ seconds, the

probabilities for the particle to be found in containers A, B are respectively Pa_n, Pb_n, then at time $t_{n+1} = n+1$ seconds, they become

$$\begin{bmatrix} Pa_{n+1} \\ Pb_{n+1} \end{bmatrix} = M \cdot \begin{bmatrix} Pa_n \\ Pb_n \end{bmatrix}$$

Prove that if $Pa_n + Pb_n = 1$, then $Pa_{n+1} + Pb_{n+1} = 1$.

e) Prove that at time $t_\infty = \lim_{n\to\infty} t_n$, the probabilities for finding the particle on either side of the wall

$$\begin{bmatrix} Pa_\infty \\ Pb_\infty \end{bmatrix} = \lim_{n\to\infty} M^n \cdot \begin{bmatrix} Pa_0 \\ Pb_0 \end{bmatrix}$$

are equal to each other, regardless of the initial probabilities Pa_0, Pb_0.

3. A particle of mass m and electric charge q enters a region of constant magnetic field $\vec{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. The particle moves according to Newton's IInd Law

$$m\frac{d\vec{v}}{dt} = \vec{F}_L = q\vec{v} \times \vec{B},$$

where $\vec{F}_L$ is the Lorentz force.

a) Use the results of problem 1 and prove that the equation can be rewritten as

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -\frac{q}{m}\begin{bmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

b) Assume a solution of the type

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = e^{\lambda t}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},$$

where λ is a constant to be determined and $\vec{w}$ is a constant vector, the velocity at time $t = 0$. Simplify the exponential factor and write the resulting system of linear equations for the unknown quantities w_1, w_2, w_3.

c) Determine for which values of λ the system of equations has nontrivial solutions.

4. Let $\vec{r} = x\mathbf{i} + y\mathbf{j}$ be a vector in the plane, corresponding to the complex number $z = re^{i\theta}$.

a) Rotating the vector $\vec{r}$ counterclockwise by an angle α leads to the vector $\vec{r'} = x'\mathbf{i} + y'\mathbf{j}$, corresponding to the complex number $z' = r'e^{i\theta'}$. Find the relations between r', θ' and r, θ.

b) Consider the columns $\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix}$ associated to these two vectors and write the relations between coordinates as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \cdot \begin{bmatrix} x \\ y \end{bmatrix},$$

where R is a 2×2 real matrix to be determined.
c) Compute the determinant of R and its inverse, R^{-1}.
d) Compute the transpose of R, R^T and prove that $R^T = R^{-1}$.
e) Using the results of problem 1 and part d) above, prove that the scalar product of two vectors $\vec{r_1}, \vec{r_2}$ is equal to the scalar product between the corresponding vectors rotated by the same angle, $R\vec{r_1}$ and $R\vec{r_2}$.

5. Consider the equation of motion for a simple harmonic oscillator

$$m\frac{d^2x}{dt^2} = -kx$$

and solve it by the follwing method:
a) Define $\omega_0^2 = \frac{k}{m}$ and rewrite the equation as

$$\frac{dv}{dt} = -\omega_0^2 x,$$

where $v = \frac{dx}{dt}$.
b) Find the matrix M in the expression

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = M \cdot \begin{bmatrix} x \\ v \end{bmatrix}$$

c) Assume a solution of the type

$$\begin{bmatrix} x \\ v \end{bmatrix}(t) = e^{\lambda t}\begin{bmatrix} x_0 \\ v_0 \end{bmatrix},$$

where x_0, v_0 are the initial position and velocity and λ is a constant to be determined.
d) Find the values of λ for which the linear system of equations in x_0, v_0 has nontrivial solutions.
e) Write the most general combination of solutions found at d).

6. Computing Potentials As known from Newtonian mechanics and from electrostatics, the potential energy of interaction (in three dimensions) between two point charges q_1 and q_2 (in mechanics, between two point masses m_1 and m_2) is given by the formula

$$W(q_1, q_2) = K\frac{q_1 q_2}{r}, \tag{1}$$

where K is a constant and r is the distance between the two points. The force associated with this energy has components

$$F_x = -\frac{\partial W}{\partial x}, \qquad F_y = -\frac{\partial W}{\partial y}, \qquad F_z = -\frac{\partial W}{\partial z} \tag{2}$$

Use the equations (1) and (2) to compute the energy and the z component of the force acting of the point particle for the following situations:

a) point charge q, situated at $(0, 0, z)$ and interacting with the infinite plane $z = 0$, with uniform charge density σ;

b) point charge q, situated at $(0, 0, z)$ and interacting with the disc $z = 0, 0 \leq r \leq R$, with uniform charge density σ;

c) point charge q, situated at $(0, 0, z)$ and interacting with the spherical shell $r = R, R < z$, with uniform charge density σ;

d) point charge q, situated at $(0, 0, z)$ and interacting with the sphere $r \leq R, R < z$, with uniform charge density ρ;

e) Is it possible to obtain as result an infinite energy and a finite force? Interpret your result.

7. Computing Moments of Inertia Find the moments of inertia relative to the x, y and z axes for the following uniform mass distributions of density 1:

a) the disc $r \leq R, z = 0$;

b) the sphere $r \leq R$;

c) the domain $-L \leq x \leq L, -W \leq y \leq W, z = 0$;

d) the cylinder $r \leq R, -H \leq z \leq H$.

8. Maxwell's Equations in Differential and Integral Forms

Our fundamental knowledge of electromagnetism can be summarized in four linear, partial differential equations, describing the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, as functions of space and time.

In differential form, these equations read:

$$\begin{cases} \vec{\nabla} \times \vec{E} & = & -\frac{\partial \vec{B}}{\partial t} & & \text{Faraday} \\ \vec{\nabla} \times \vec{B} & = & \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} & +\mu_0 \vec{j} & \text{Ampère} \\ \vec{\nabla} \cdot \vec{E} & = & \frac{\rho}{\epsilon_0} & & \text{Gauss} \\ \vec{\nabla} \cdot \vec{B} & = & 0, & & \end{cases}$$

and are supplemented by the equation of continuity

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0.$$

The Ampère and Gauss equations are also referred to as the equations with sources, because they indicate the dependence of the electric and magnetic fields on the sources of these fields, the electric charge density ρ and the electric current density $\vec{j}$, respectively.

a) Rewrite Maxwell's equations for static fields (all time derivatives are zero).

b) Using the divergence theorem, prove that the Gauss law has the integral form

$$\Phi_e = \oint_S \vec{E} \cdot d\vec{S} = \frac{Q_V}{\epsilon_0},$$

where Φ_e is the electric field flux though the closed surface S, enclosing the volume V, and Q_V is the total electric charge in the volume V.

Use the integral form and symmetry arguments to compute the electric field produced by the following charge densities:

(i) point charge q, placed at the origin, in three dimensions; (ii) point charge q, placed at the origin, in two dimensions; (iii) point charge q, placed at the origin, in one dimension; (iv) sphere of charge Q, with center at the origin, in three dimensions.

c) Use the divergence theorem for the last of the four equations and show (by analogy with (b)) that magnetic point charges (magnetic monopoles) do not exist.

d) Using Stoke's theorem, prove that the Ampère law for static fields has the integral form

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot d\vec{S} = \mu_0 I_S,$$

where C is any closed curve, and S any surface that has C as boundary. I_S is the total current flowing through the surface S.

Use the integral form and symmetry arguments to compute the magnetic field produced by a current of density

$$\vec{j}(r) = r\mathbf{k},$$

flowing along an infinitelly long cylinder of radius R, parallel to the z axis.

e) Using Stoke's theorem, prove that the Faraday law has the integral form

$$\frac{d\Phi_m}{dt} = -\oint_C \vec{E} \cdot d\vec{l},$$

where Φ_m is the magnetic field flux though the surface S, of boundary the closed curve C.

f) Prove that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$. Use this result and the equation of continuity to show that the constant c is the speed of light:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}.$$

9. Electromagnetic Potentials

Consider Maxwell's equations for static fields (all time derivatives are zero).

a) Show that the Faraday equation proves that the electrostatic field $\vec{E}$ is conservative and it can be derived from a scalar function, the electrostatic potential ϕ:

$$\vec{E} = -\vec{\nabla}\phi.$$

b) Prove using the Maxwell equation that the magnetic field $\vec{B}$ can be derived from a vector function, the vector potential $\vec{A}$:

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

c) Use the Gauss law and show that the electrostatic potential satisfies the Poisson equation

$$\Delta\phi = (\vec{\nabla})^2\phi = -\frac{\rho}{\epsilon_0}.$$

d) Use the Ampère law and show that the components of the vector potential satisfy the Poisson equation

$$\Delta(\vec{A})_i = (\vec{\nabla})^2(\vec{A})_i = -\mu_0(\vec{j})_i,$$

where $i = x, y, z$, provided that $\vec{\nabla} \cdot \vec{A} = 0$.

e) Using the definitions, identify the conservative fields and find their corresponding potentials:

(i) $x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$; (ii) $y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$; (iii) $\hat{\mathbf{i}}$; (iv) $\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\mathbf{k}}{r^3}$.

f) Using the definition, identify the zero divergence fields and find their vector potentials:

(i) $x\hat{\mathbf{i}} - y\hat{\mathbf{j}}$; (ii) $y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$; (iii) $\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

g) Assume that the sources are zero in the equations for static fields. Is it possible to have non-zero fields without sources? Can the potentials be non-vanishing without sources?

10. Electromagnetic Waves

Consider the time dependent Maxwell equations, where all the sources are zero.

a) Prove that all components of both the electric and the magnetic fields satisfy the wave equation

$$\Delta f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0.$$

b) Assume that the electric field has only one component, along x, and that the magnetic field has only one component, along y. Identify which of the fields are solutions to the wave equation and find the magnetic field that corresponds to each of them:

(i) $\vec{E} = \cos(z + ct)\hat{\mathbf{i}}$; (ii) $\vec{E} = \sin(z - ct)\hat{\mathbf{i}}$; (iii) $\vec{E} = \cos^3(z)\hat{\mathbf{i}}$; (iv) $\vec{E} = zct\hat{\mathbf{i}}$.

c) Is it possible to have an electromagnetic wave propagating along the z direction without any charges or currents being present?

11. Electromagnetic Static Field Energy

Consider again Maxwell's equations for static fields

$$\begin{cases} \vec{\nabla} \times \vec{E} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{j} \\ \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{cases},$$

together with the definitions of potentials

$$\vec{E} = -\vec{\nabla}\Phi, \qquad \vec{B} = \vec{\nabla} \times \vec{A}.$$

a) Prove the identity

$$\vec{\nabla}(\Phi\vec{E}) = \frac{\rho\Phi}{\epsilon_0} - \vec{E} \cdot \vec{E}.$$

Using this identity, show that

$$\frac{1}{\epsilon_0}\int_V \rho\Phi d\tau = \int_V |\vec{E}|^2 d\tau + \oint_S \Phi\vec{E} \cdot d\vec{\sigma},$$

where V is any volume and S is the closed surface that bounds V.

The total energy of the electrostatic field is given by

$$W_e = \frac{1}{2}\int_{\mathbf{R}^3} \rho\Phi d\tau,$$

and the integral is extended to the whole space. Assuming that the surface integral in the previous result is zero in this limit, show that the total energy of the electrostatic field is

$$W_e = \frac{\epsilon_0}{2}\int_{\mathbf{R}^3} |\vec{E}|^2 d\tau.$$

Use this result to prove that if the charge density is zero everywhere, the electric field $\vec{E}$ must also vanish everywhere (no electrostatic field can exist without electric charges).

b) Demonstrate that

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \vec{B} - \mu_0 \vec{A} \cdot \vec{j}.$$

Using this identity, show that

$$\int_V \vec{j} \cdot \vec{A} d\tau = \frac{1}{\mu_0}\int_V |\vec{B}|^2 d\tau + \oint_S \vec{A} \times \vec{B} \cdot d\vec{\sigma},$$

where V is any volume and S is the closed surface that bounds V.

The total energy of the magnetostatic field is given by

$$W_m = \frac{1}{2}\int_{\mathbf{R}^3} \vec{j}\cdot\vec{A}d\tau,$$

and the integral is extended to the whole space. Assuming that the surface integral in the previous result is zero in this limit, show that the total energy of the magnetostatic field is

$$W_m = \frac{1}{2\mu_0}\int_{\mathbf{R}^3} |\vec{B}|^2 d\tau.$$

Use this result to prove that if the current density is zero everywhere, the magnetic field $\vec{B}$ must also vanish everywhere (no magnetostatic field can exist without electric currents).

12. Solving Maxwell's Equations

a) Consider the complete Maxwell's equations for vacuum

$$\begin{cases} \vec{\nabla}\times\vec{E} & = & -\frac{\partial\vec{B}}{\partial t} & \\ \vec{\nabla}\times\vec{B} & = & \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} & +\mu_0\vec{j} \\ \vec{\nabla}\cdot\vec{E} & = & \frac{\rho}{\epsilon_0} & \\ \vec{\nabla}\cdot\vec{B} & = & 0, & \end{cases}$$

and assume a solution of the type

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla}\times\vec{A}.$$

Prove that this substitution solves the equations without sources.

b) Use the substitution in the equations with sources and show that they become

$$\begin{cases} \Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} & = & -\mu_0\vec{j} & +\vec{\nabla}(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}) \\ \Delta\Phi + \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial t} & = & -\frac{\rho}{\epsilon_0}. & \end{cases}$$

c) Impose the condition (called *gauge condition*)

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$$

and show that the equations with sources become

$$\begin{cases} \Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} & = & -\mu_0\vec{j} \\ & & \\ \Delta\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} & = & -\frac{\rho}{\epsilon_0}. \end{cases}$$

Show that for static fields these equations become Poisson equations.

13. Electromagnetic Waves and Fourier Analysis

In homework set 8 we showed that in the absence of sources, the equations satisfied by the electric and magnetic fields are

$$\begin{cases} \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0, \end{cases}$$

and they lead to the wave equations

$$\Delta \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0, \qquad \Delta \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0.$$

In the following, we assume that all the fields are functions of z and t only (no x, y dependence).

a) Prove that if the electric field has only one nonzero component, along x, then the magnetic field has only one nonzero component, along y.

b) Solve the wave equation for $E_x(z,t)$ subject to the initial condition $E_x(z,0) = \cos(kz)$ and boundary condition $E_x(0,t) = \cos(\omega t)$, where ω and k are given constants.

c) Find the magnetic field $B_y(z,t)$ corresponding to the electric field in b).

Review Questions

14. Eigenvectors, Eigenvalues and Systems of Linear Equations

a) Consider the equation of motion (for a harmonic oscillator with friction)

$$\frac{d^2x}{dt^2} = -3\frac{dx}{dt} - 2x.$$

(i). Denote $\frac{dx}{dt} = v(t)$ and rewrite the equation in the form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{bmatrix} = A \cdot \begin{bmatrix} x \\ v \end{bmatrix},$$

where A is a 2×2 matrix of real numbers to be determined.

(ii). Assume that the solution has the form

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix},$$

where x_0, v_0 are constants, and find all the possible values of λ (the eigenvalues) for which the system of equations has nontrivial solutions.

(iii). Find the eigenvectors corresponding to the eigenvalues determined in (ii).

b) Consider the equation (for rotations in the plane)

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$

(i). Assume that the solution has the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix},$$

where x_0, y_0 are constants, and find all the possible values of λ (the eigenvalues) for which the system of equations has nontrivial solutions.

(ii). Find the eigenvectors corresponding to the eigenvalues determined in (i).

(iii). Using the equation, prove that for any value of t,

$$\begin{bmatrix} x(t) & y(t) \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = 0,$$

and from here, that

$$\frac{d}{dt}\left[x^2(t) + y^2(t)\right] = 0.$$

Interpret the result.

15. Partial Differentiation Consider the differential equation

$$\frac{\partial^2 f}{\partial x^2} + 3\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0.$$

(i). Perform the change of variables

$$\begin{cases} x &= r + s \\ y &= r + 2s \end{cases}$$

and rewrite the equation in the new variables.

(ii). Identify which ones of the following functions are solutions for this equation:

A) $f(x, y) = \cos^3(x)$ B) $f(x, y) = e^{y-x} + \sin^2(2x - y)$ C) $f(x, y) = \ln[(2x - y)(y - x)]$.

16. Lagrange Multipliers

a) Find the distance from the origin to the plane given by the equation

$$\frac{x}{m}+\frac{y}{n}+\frac{z}{p}=1,$$

where m, n, p are non-zero constants. The distance from the origin to a plane is given by the minimum of the function $f(x,y,z)=\sqrt{x^2+y^2+z^2}$, subject to the constraint equation describing the plane.

b) Find the minimum value of the energy

$$W(I,V)=\frac{CV^2}{2}+\frac{RI^2}{2},$$

subject to the average power constraint

$$P=IV=\text{constant},$$

where R, C, P are constants.

17. Infinite Series

a) Find

$$\lim_{x\to 0}\frac{\sin x}{x}$$

b) Compute

$$F_N(q)=1+e^{iq}+e^{2iq}+\ldots+e^{i(N-1)q},$$

where q is a real number and $N>1$.

c) Find the absolute value of $F_N(q)$ and use it to derive

$$\lim_{q\to 0}|F_N(q)|$$

d) Are the limits $\lim\limits_{N\to\infty}\lim\limits_{q\to 0}|F_N(q)|$ and $\lim\limits_{q\to 0}\lim\limits_{N\to\infty}|F_N(q)|$ equal to each other?

e) Find the sum of the series

$$\sum_{n=3}^{\infty}\frac{4}{n^2-4}$$

17. a) Compute

$$\lim_{x\to 0}\left[\frac{6}{\sin^2(x)}+\frac{\ln(1-x)}{x-\sin(x)}\right]$$

b) Use Euler's formula to prove that

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta).$$

for any real numbers α, β.

18. a) Find the distance from the origin to the plane given by the equation $x + y + z = 1$.

b) Find the values of the real number a for which the following system of equations has a unique solution:

$$\begin{cases} x + ay = 1 \\ ax + y = 1 \\ x + y = a \end{cases}$$

19. a) Prove that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be parametrized by

$$\begin{cases} x = a\cos(t) \\ y = b\sin(t), \end{cases}$$

where $t \in [0, 2\pi)$.

b) Compute the area of the ellipse given at point a).

20. a) Compute the line integral $\oint \vec{F} \cdot d\vec{r}$ of the vector $\vec{F}(\vec{r}) = \hat{\mathbf{k}} \times \vec{r}$ along the circle $(x-1)^2 + (y-1)^2 = 1$.

b) Compute the surface integral $\oint \vec{F} \cdot d\vec{\sigma}$ of the vector $\vec{F}(\vec{r}) = \vec{r}$ through the surface of the cylinder $0 \leq z \leq 1, 0 \leq x^2 + y^2 \leq 1$.

21. Find the Fourier series of the function

$$f(x) = \left(x - \frac{\pi}{2}\right)^2, x \in [-\pi, \pi).$$